ON f-HARMONIC MORPHISMS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract

f-Harmonic maps were first introduced and studied by Lichnerowicz in [18] (see also Section 10.20 in Eells-Lemaire's report [10]). In this paper, we study a subclass of f-harmonic maps called f-harmonic morphisms which pull back local harmonic functions to local f-harmonic functions. We prove that a map between Riemannian manifolds is an f-harmonic morphism if and only if it is a horizontally weakly conformal f-harmonic map. This generalizes the well-known Fuglede-Ishihara characterization for harmonic morphisms. Some properties and many examples as well as some non-existence of f-harmonic morphisms are given. We also study the f-harmonicity of conformal immersions.

1. f-Harmonic maps vs. F-harmonic maps

1.1 f-harmonic maps

Let $f : (M, g) \longrightarrow (0, \infty)$ be a smooth function. An *f-harmonic map* is a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds such that $\phi|_\Omega$ is a critical point of the f-energy (see [18], and [10], Section 10.20)

$$E_f(\phi) = \frac{1}{2}\int_\Omega f\,|\mathrm{d}\phi|^2 dv_g,$$

for every compact domain $\Omega \subseteq M$. The Euler-Lagrange equation gives the f-harmonic map equation ([5], [25])

$$\tau_f(\phi) \equiv f\tau(\phi) + \mathrm{d}\phi(\mathrm{grad}\, f) = 0, \tag{1}$$

where $\tau(\phi) = \mathrm{Tr}_g \nabla\, d\phi$ is the tension field of ϕ vanishing of which means ϕ is a harmonic map.

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Example 1. Let $\varphi,\ \psi,\ \phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined as

$$\begin{aligned} \varphi(x,y,z) &= (x,y), \\ \psi(x,y,z) &= (3x, xy), \quad \text{and} \\ \phi(x,y,z) &= (x, y+z). \end{aligned}$$

Then, one can easily check that both φ and ψ are f-harmonic map with $f = e^z$, φ is a horizontally conformal submersion whilst ψ is not. Also, ϕ is an f-harmonic map with $f = e^{y-z}$, which is a submersion but not horizontally weakly conformal.

1.2 F-harmonic map

Let $F : [0,+\infty) \longrightarrow [0,+\infty)$ be a C^2-function, strictly increasing on $(0,+\infty)$, and let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between Riemannian manifolds. Then φ is said to be an *F-harmonic map* if $\varphi|_\Omega$ is a critical point of the F-energy functional

$$E_F(\varphi) = \int_\Omega F(\frac{|d\varphi|^2}{2})\, v_g,$$

for every compact domain $\Omega \subseteq M$. The equation of F-harmonic maps is given by ([2])

$$\tau_F(\varphi) \equiv F'(\frac{|d\varphi|^2}{2})\tau(\varphi) + \varphi_*\left(\operatorname{grad} F'(\frac{|d\varphi|^2}{2})\right) = 0, \tag{2}$$

where $\tau(\varphi)$ denotes the tension field of φ.

Harmonic maps, *p-harmonic maps*, and *exponential harmonic maps* are examples of F-harmonic maps with $F(t) = t$, $F(t) = \frac{1}{p}(2t)^{p/2}$ $(p > 4)$, and $F(t) = e^t$ respectively ([2]).

In particular, p-harmonic map equation can be written as

$$\tau_p(\varphi) = |d\varphi|^{p-2}\tau(\varphi) + d\varphi(\operatorname{grad}|d\varphi|^{p-2}) = 0, \tag{3}$$

1.3 Relationship between f-harmonic and F-harmonic maps

We can see from Equation (1) that an f-harmonic map with f = constant > 0 is nothing but a harmonic map so both f-harmonic maps and F-harmonic maps are generalizations of harmonic maps. Though we were warned in [5] that f-harmonic maps should not be confused with F-harmonic maps and p-harmonic maps, we observe that, apart from critical points, any F-harmonic map is a special f-harmonic maps. More precisely we have

Corollary 1.1. *Any F-harmonic map $\varphi : (M,g) \longrightarrow (N,h)$ without critical points, i.e., $|d\varphi_x| \neq 0$ for all $x \in M$, is an f-harmonic map with $f = F'(\frac{|d\varphi|^2}{2})$.*

In particular, a p-harmonic map without critical points is an f-harmonic map with $f = |d\varphi|^{p-2}$

Proof. Since F is a C^2-function and strictly increasing on $(0, +\infty)$ we have $F'(t) > 0$ on $(0, +\infty)$. If the F-harmonic map $\varphi : (M, g) \longrightarrow (N, h)$ has no critical points, i.e., $|d\varphi_x| \neq 0$ for all $x \in M$, then the function $f : (M, g) \longrightarrow (0, +\infty)$ with $f = F'(\frac{|d\varphi|^2}{2})$ is a smooth and we see from Equations (2) and (1) that the F-harmonic map φ is an f-harmonic map with $f = F'(\frac{|d\varphi|^2}{2})$. The second statement follows from the fact that for a p-harmonic map, $F(t) = \frac{1}{p}(2t)^{p/2}$ and hence $f = F'(\frac{|d\varphi|^2}{2}) = |d\varphi|^{p-2}$.

□

Another relationship between f-harmonic maps and harmonic maps can be characterized as follows.

Corollary 1.2. *A map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is f-harmonic if and only if $\phi : (M^m, f^{\frac{2}{m-2}} g) \longrightarrow (N^n, h)$ is a harmonic map.*

Proof. The statement that the f-harmoninicity of $\phi : (M^m, g) \longrightarrow (N^n, h)$ implies the harmonicity of $\phi : (M^m, f^{\frac{2}{m-2}} g) \longrightarrow (N^n, h)$ was stated and proved in [18] (see also Section 10.20 in [10]). It is not difficult to see that the converse is also true. In fact, let $\bar{g} = f^{\frac{2}{m-2}} g$, a straightforward computation shows that

$$\tau(\varphi, \bar{g}) = f^{\frac{-m}{m-2}} \tau_f(\varphi, g),$$

which completes the proof of the corollary. □

1.4 A physical motivation for the study of f-harmonic maps: In physics, the equation of motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction (such a model is called the inhomogeneous Heisenberg ferromagnet) is given by

$$\frac{\partial u}{\partial t} = f(x)(u \times \Delta u) + \nabla f \cdot (u \times \nabla u), \tag{4}$$

where $\Omega \subseteq \mathbb{R}^m$ is a smooth domain in the Euclidean space, f is a real-valued function defined on Ω, $u(x, t) \in S^2$, $\times$ denotes the cross products in $\mathbb{R}^3$ and Δ is the Laplace operator on $\mathbb{R}^m$. Physically, the function f is called the coupling function, and is the continuum limit of the coupling constants between the neighboring spins. Since u is a map into S^2 it is well known that the tension field of u can be written as $\tau(u) = \Delta u + |\nabla u|^2 u$, and one can easily check that the right hand side of the inhomogeneous Heisenberg spin system (4) can be written as $u \times (f\tau(u) + \nabla f \cdot \nabla u)$. It follows that u is a smooth stationary solution of (4) if and only if $f\tau(u) + \nabla f \cdot \nabla u = 0$, i.e., u is an f-harmonic map. So

there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (4) on the domain Ω and the set of f-harmonic maps from Ω into 2-sphere. The above inhomogeneous Heisenberg spin system (4) is also called inhomogeneous Landau-Lifshitz system (see, e.g., [6], [7], [9], [14], [16], [17], for more details).

Using Corollary 1.2 we have the following example which provides many stationary solutions of the inhomogeneous Heisenberg spin system defined on $\mathbb{R}^3$.

Example 2. $u : (\mathbb{R}^3, ds_0) \longrightarrow (N^n, h)$ is an f-harmonic map if and only if $u : (\mathbb{R}^3, f^2 ds_0) \longrightarrow (N^n, h)$ is a harmonic map. In particular, there is a 1-1 correspondence between harmonic maps from 3-sphere $S^3 \setminus \{N\} \equiv (\mathbb{R}^3, \frac{4ds_0}{(1+|x|^2)^2}) \longrightarrow (N^n, h)$ and f-harmonic maps with $f = \frac{2}{1+|x|^2}$ from Euclidean 3-space $\mathbb{R}^3 \longrightarrow (N^n, h)$. When $(N^n, h) = S^2$, we have a $1-1$ correspondence between the set of harmonic maps $S^3 \longrightarrow S^2$ and the set of stationary solutions of the inhomogeneous Heisenberg spin system on $\mathbb{R}^3$. Similarly, there is a 1-1 correspondence between harmonic maps from hyperbolic 3-space $H^3 \equiv (D^3, \frac{4ds_0}{(1-|x|^2)^2}) \longrightarrow (N^n, h)$ and f-harmonic maps $(D^3, ds_0) \longrightarrow (N^n, h)$ with $f = \frac{2}{1-|x|^2}$ from unit disk in Euclidean 3-space.

1.5 A little more about f-harmonic maps

Corollary 1.3. *If $\phi : (M^m, g) \longrightarrow (N^n, h)$ is an f_1-harmonic map and also an f_2-harmonic map, then* $\mathrm{grad}(f_1/f_2) \in \ker \mathrm{d}\phi$.

Proof. This follows from

$$\tau_{f_1}(\phi) \equiv f_1 \tau(\phi) + \mathrm{d}\phi(\mathrm{grad}\, f_1) = 0,$$
$$\tau_{f_2}(\phi) \equiv f_2 \tau(\phi) + \mathrm{d}\phi(\mathrm{grad}\, f_2) = 0,$$

and hence

$$\mathrm{d}\phi(\mathrm{grad} \ln (f_1/f_2)) = 0.$$

□

Proposition 1.4. *A conformal immersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ with $\phi^* h = \lambda^2 g$ is f-harmonic if and only if it is m-harmonic and $f = C\lambda^{m-2}$. In particular, an isometric immersion is f-harmonic if and only if $f = const$ and hence it is harmonic.*

Proof. It is not difficult to check (see also [26]) that for a conformal immersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ with $\phi^* h = \lambda^2 g$, the tension field is given by

$$\tau(\phi) = m\lambda^2 \eta + (2-m)\mathrm{d}\phi\, (\mathrm{grad} \ln\lambda)\,,$$

so we can compute the f-tension field to have

$$\tau_f(\phi) = f[m\lambda^2\eta + \mathrm{d}\phi\left(\mathrm{grad}\ln(\lambda^{2-m}f)\right)],$$

where η is the mean curvature vector of the submanifold $\phi(M) \subset N$. Noting that η is normal part whilst $\mathrm{d}\phi\,(\mathrm{grad}\ln\lambda^{2-m}f)$ is the tangential part of $\tau_f(\phi)$ we conclude that $\tau_f(\phi) = 0$ if and only if

$$\begin{cases} m\lambda^2\eta = 0, \\ \mathrm{d}\phi\,(\mathrm{grad}\ln(\lambda^{2-m}f)) = 0. \end{cases}$$

It follows that $\eta = 0$ and $\mathrm{grad}\,(\ln(\lambda^{2-m}f)) = 0$ since ϕ is an immersion. From these we see that ϕ is a minimal conformal immersion which means it is an m-harmonic map ([26]) and that $f = C\lambda^{m-2}$. Thus, we obtain the first statement. The second statement follows from the first one with $\lambda = 1$. □

2. f-Harmonic morphisms

A *horizontally weakly conformal* map is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds such that for each $x \in M$ at which $\mathrm{d}\varphi_x \neq 0$, the restriction $\mathrm{d}\varphi_x|_{H_x} : H_x \longrightarrow T_{\varphi(x)}N$ is conformal and surjective, where the horizontal subspace H_x is the orthogonal complement of $V_x = \ker \mathrm{d}\varphi_x$ in T_xM. It is not difficult to see that there is a number $\lambda(x) \in (0, \infty)$ such that $h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda^2(x)g(X, Y)$ for any $X, Y \in H_x$. At the point $x \in M$ where $\mathrm{d}\varphi_x = 0$ one can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \longrightarrow R$ which is called the *dilation* of a horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is called *horizontally homothetic* if the gradient of $\lambda^2(x)$ is vertical meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field X on M. Recall that a C^2 map $\varphi : (M, g) \longrightarrow (N, h)$ is a *p-harmonic morphism* ($p > 1$) if it preserves the solutions of p-Laplace equation in the sense that for any p-harmonic function $f : U \longrightarrow \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \longrightarrow \mathbb{R}$ is a p-harmonic function. A p-harmonic morphism can be characterized as a horizontally weakly conformal p-harmonic map (see [11], [15], [19] for details).

Definition 2.1. *Let $f : (M, g) \longrightarrow (0, \infty)$ be a smooth function. A C^2-function $u : U \longrightarrow \mathbb{R}$ defined on an open subset U of M is called f-harmonic if*

$$\Delta_f^M u \equiv f\Delta^M u + g(\mathrm{grad} f, \mathrm{grad}\, u) = 0. \tag{5}$$

A continuous map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is called an f-harmonic morphism if for every harmonic function u defined on an open subset V of N such that $\phi^{-1}(V)$ is non-empty, the composition $u \circ \phi$ is f-harmonic on $\phi^{-1}(V)$.

Theorem 2.2. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. Then, the following are equivalent:*
(1) *ϕ is an f-harmonic morphism;*
(2) *ϕ is a horizontally weakly conformal f-harmonic map;*
(3) *There exists a smooth function λ^2 on M such that*

$$\Delta_f^M (u \circ \phi) \;=\; f\lambda^2 (\Delta^N u) \circ \phi$$

for any C^2-function u defined on (an open subset of) N.

Proof. We will need the following lemma to prove the theorem.

Lemma 2.3. ([15]) *For any point $q \in (N^n, h)$ and any constants C_σ, $C_{\alpha\beta}$ with $C_{\alpha\beta} = C_{\beta\alpha}$ and $\sum_{\alpha=1}^n C_{\alpha\alpha} = O$, there exists a harmonic function u on a neighborhood of q such that $u_\sigma(q) = C_\sigma, u_{\alpha\beta}(q) = C_{\alpha\beta}$.*

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a map and let $p \in M$. Suppose that $\phi(x) = (\phi^1(x), \phi^2(x), \cdots, \phi^n(x))$ is the local expression of ϕ with respect to the local coordinates $\{x^i\}$ in the neighborhood $\phi^{-1}(V)$ of p and $\{y^\alpha\}$ in a neighborhood V of $q = \phi(p) \in N$. Let $u : V \longrightarrow \mathbb{R}$ defined on an open subset V of N. Then, a straightforward computation gives

$$\begin{aligned} \Delta_f^M (u \circ \phi) &= f\Delta^M (u \circ \phi) + d(u \circ \phi)(\mathrm{grad} f) \\ &= f u_{\alpha\beta} g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) + f u_\alpha \Delta^M \phi^\alpha + d(u \circ \phi)(\mathrm{grad} f) \\ &= f g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) u_{\alpha\beta} + [f\, \Delta^M \phi^\sigma + (\mathrm{grad} f)\phi^\sigma] u_\sigma. \end{aligned} \tag{6}$$

By Lemma 2.3, we can choose a local harmonic function u on $V \subset N$ such that $u_\sigma(q) = C_\sigma = 0\ \forall\ \sigma = 1, 2, \cdots, n,\ u_{\alpha\beta}(q) = 1\ \ (\alpha \neq \beta)$, and all other $u_{\rho\sigma}(q) = C_{\rho\sigma} = 0$ and substitute it into (6) to have

$$g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) = 0,\ \forall\ \alpha \neq \beta = 1, 2, \cdots, n. \tag{7}$$

Note that the choice of such functions implies

$$h^{\alpha\beta}(\phi(p)) = 0,\ \forall\ \alpha \neq \beta = 1, 2, \cdots, n. \tag{8}$$

Another choice of harmonic function u with $C_{11} = 1,\ C_{\alpha\alpha} = -1\ \ (\alpha \neq 1)$ and all other $C_\sigma, C_{\alpha\beta} = 0$ for Equation (6) gives

$$g(\mathrm{grad}\phi^1, \mathrm{grad}\phi^1) - g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\alpha) = 0,\ \forall\ \alpha \neq \beta = 2, 3, \cdots, n. \tag{9}$$

Note also that for these choices of harmonic functions u we have

$$h^{11}(\phi(p)) - h^{\alpha\alpha}(\phi(p)) = 0,\ \forall\ \alpha \neq \beta = 2, 3, \cdots, n. \tag{10}$$

It follows from (7), (8), (9) and (10) that the f-harmonic morphism ϕ is a horizontally weakly conformal map

$$g(\mathrm{grad}\phi^{\alpha}, \mathrm{grad}\phi^{\beta}) = \lambda^2 h^{\alpha\beta} \circ \phi. \tag{11}$$

Substituting horizontal conformality equation (11) into (6) we have

$$\begin{aligned} \Delta_f^M(u \circ \phi) &= f\lambda^2(h^{\alpha\beta} \circ \phi)u_{\alpha\beta} + [f\,\Delta^M \phi^{\sigma} + (\mathrm{grad} f)\phi^{\sigma}]u_{\sigma} \\ &= f\lambda^2(\Delta^N u) \circ \phi + [f\,\Delta^M \phi^{\sigma} + f\lambda^2(h^{\alpha\beta}\bar{\Gamma}^{\sigma}_{\alpha\beta}) \circ \phi + (\mathrm{grad} f)\phi^{\sigma}]u_{\sigma} \\ &= f\lambda^2(\Delta^N u) \circ \phi + \mathrm{d}u\,(\tau_f(\phi)) \end{aligned} \tag{12}$$

for any function u defined (locally) on N. By special choice of harmonic function u we conclude that the f-harmonic morphism is an f-harmonic map. Thus, we obtain the implication "(1) $\Longrightarrow$ (2)". Note that the only assumption we used to obtain Equation (12) is the horizontal conformality (11). Therefore, it follows from (12) that "(2) $\Longrightarrow$ (3)". Finally, "(3) $\Longrightarrow$ (1)" is clearly true. Thus, we complete the proof of the theorem. □

Similar to harmonic morphisms we have the following regularity result.

Corollary 2.4. *For $m \geq 3$, an f-harmonic morphism $\phi : (M^m, g) \longrightarrow (N^n, h)$ is smooth.*

Proof. In fact, by Corollary 1.1, if $m \neq 2$ and $\phi : (M^m, g) \longrightarrow (N^n, h)$ is an f-harmonic morphism, then $\phi : (M^m, f^{2/(m-2)}g) \longrightarrow (N^n, h)$ is a harmonic map and hence a harmonic morphism, which is known to be smooth (see, e. g., [4]). □

It is well known that the composition of harmonic morphisms is again a harmonic morphism. The composition law for f-harmonic morphisms, however, will need to be modified accordingly. In fact, by the definitions of harmonic morphisms and f-harmonic morphisms we have

Corollary 2.5. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be an f-harmonic morphism with dilation λ_1 and $\psi : (N^n, h) \longrightarrow (Q^l, k)$ a harmonic morphism with dilation λ_2. Then the composition $\psi \circ \phi : (M^m, g) \longrightarrow (Q^l, k)$ is an f-harmonic morphism with dilation $\lambda_1(\lambda_2 \circ \phi)$.*

More generally, we can prove that f-harmonic morphisms pull back harmonic maps to f-harmonic maps.

Proposition 2.6. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be an f-harmonic morphism with dilation λ and $\psi : (N^n, h) \longrightarrow (Q^l, k)$ a harmonic map. Then the composition $\psi \circ \phi : (M^m, g) \longrightarrow (Q^l, k)$ is an f-harmonic map.*

Proof. It is well known (see e.g., [4], Proposition 3.3.12) that the tension field of the composition map is given by

$$\tau(\psi\circ\phi) = \mathrm{d}\psi(\tau(\phi)) + \mathrm{Tr}_g\nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi),$$

from which we have the f-tension of the composition $\psi\circ\phi$ given by

$$\tau_f(\psi\circ\phi) = \mathrm{d}\psi(\tau_f(\phi)) + f\mathrm{Tr}_g\nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi). \tag{13}$$

Since ϕ is an f-harmonic morphism and hence a horizontally weakly conformal f-harmonic map with dilation λ, we can choose a local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ around $p \in M$ and $\{\epsilon_1, \ldots, \epsilon_n\}$ around $\phi(p) \in N$ so that

$$\begin{cases} \mathrm{d}\phi(e_i) = \lambda\epsilon_i, & i = 1, \ldots, n, \\ \mathrm{d}\phi(e_\alpha) = 0, & \alpha = n+1, \ldots, m. \end{cases}$$

Using these local frames we compute

$$\begin{aligned} \mathrm{Tr}_g\nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi) &= \sum_{i=1}^{m} \nabla \mathrm{d}\psi(\mathrm{d}\phi e_i, \mathrm{d}\phi e_i) = \lambda^2 \left(\sum_{i=1}^{n} \nabla \mathrm{d}\psi(\epsilon_i, \epsilon_i)\right)\circ\phi \\ &= \lambda^2\tau(\psi)\circ\phi. \end{aligned}$$

Substituting this into (13) we have

$$\begin{aligned} \tau_f(\psi\circ\phi) &= f\mathrm{d}\psi(\tau(\phi)) + f\lambda^2\tau(\psi)\circ\phi + \mathrm{d}(\psi\circ\phi)(\mathrm{grad}\, f) \\ &= \mathrm{d}\psi(\tau_f(\phi)) + f\lambda^2\tau(\psi)\circ\phi, \end{aligned}$$

from which the proposition follows. □

Theorem 2.7. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally weakly conformal map with $\varphi^* h = \lambda^2 g|_{\mathcal{H}}$. Then, any two of the following conditions imply the other one.*

(1) *ϕ is an f-harmonic map and hence an f-harmonic morphism;*
(2) $\mathrm{grad}(\, f\lambda^{2-n})$ *is vertical;*
(3) *ϕ has minimal fibers.*

Proof. It can be check (see e.g., [4]) that the tension field of a horizontally weakly conformal map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is given by

$$\tau(\phi) = -(m-n)d\phi(\mu) + (2-n)d\phi(\mathrm{grad}\ln\lambda),$$

where λ is the dilation of the horizontally weakly conformal map ϕ and μ is the mean curvature vector field of the fibers. It follows that the f-tension field of ϕ can be written as

$$\tau_f(\phi) = -(m-n)fd\phi(\mu) + fd\phi(\mathrm{grad}\ln\lambda^{2-n}) + d\phi(\mathrm{grad}\, f),$$

or, equivalently,

$$\tau_f(\phi) = f[-(m-n)d\phi(\mu) + d\phi(\mathrm{grad}\ln(f\lambda^{2-n}))] = 0.$$

From this we obtain the theorem. □

An immediate consequence is the following

Corollary 2.8. (a) *A horizontally homothetic map (in particular, a Riemannian submersion) $\phi : (M^m, g) \longrightarrow (N^n, h)$ is an f-harmonic morphism if and only if $-(m-n)\mu + \mathrm{grad}\ln f$ is vertical;*
(b) *A weakly conformal map $\phi : (M^m, g) \longrightarrow (N^m, h)$ with conformal factor λ of same dimension spaces is f-harmonic and hence an f-harmonic morphism if and only if $f = C\lambda^{m-2}$ for some constant $C > 0$;*
(c) *An horizontally weakly conformal map $\phi : (M^m, g) \longrightarrow (N^2, h)$ is an f-harmonic map and hence an f-harmonic morphism if and only if $-(m-2)\mu + \mathrm{grad}\ln f$ is vertical.*

Using the characterizations of f-harmonic morphisms and p-harmonic morphisms and Corollary 1.1 we have the following corollary which provides many examples of f-harmonic morphisms.

Corollary 2.9. *A map $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds is a p-harmonic morphism without critical points if and only if it is an f-harmonic morphism with $f = |\mathrm{d}\phi|^{\mathrm{p}-2}$.*

Example 3. Möbius transformation $\phi : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{0\}$ defined by

$$\phi(x) = a + \frac{r^2}{|x-a|^2}(x-a)$$

is an f-harmonic morphism with $f(x) = C(\frac{r}{|x-a|})^{2(m-2)}$. In fact, it is well known that the Möbius transformation is a conformal map between the same dimensional spaces with the dilation $\lambda = \frac{r^2}{|x-a|^2}$. It follows from [20] that ϕ is an m-harmonic morphism, and hence by Corollary 2.9, the inversion is an f-harmonic morphism with $f = |\mathrm{d}\phi|^{\mathrm{m}-2} = (\sqrt{\mathrm{m}}\lambda)^{\mathrm{m}-2} = \mathrm{C}(\frac{\mathrm{r}}{|\mathrm{x}-\mathrm{a}|})^{2(\mathrm{m}-2)}$.

The next example is an f-harmonic morphism that does not come from a p-harmonic morphism,

Example 4. The map from Euclidean 3-space into hyperbolic plane $\phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, ds_0^2) \longrightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2)$ with $\phi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$ is an f-harmonic morphism with $f = 1/z$. Similarly, we know from [12] that the map $\phi : H^3 \equiv (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2) \longrightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2)$ with $\phi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$ is a harmonic morphism. It follows from Example 2

that the map from Euclidean space into hyperbolic plane $\phi : (\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+, ds_0^2) \longrightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2)$ with $\phi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$ is an f-harmonic map with $f = 1/z$. Since this map is also horizontally weakly conformal it is an f-harmonic morphism by Theorem 2.2.

Example 5. Any harmonic morphism $\phi : (M^m, g) \longrightarrow (N^n, h)$ is an f-harmonic morphism for a positive function f on M with vertical gradient, i.e., $d\phi(\mathrm{grad} f) = 0$. In particular, the radial projection $\phi : \mathbb{R}^{m+1} \setminus \{0\} \longrightarrow S^m,\ \ \phi(x) = \frac{x}{|x|}$ is an f-harmonic morphism for $f = \alpha(|x|)$, where $\alpha : (0, \infty) \longrightarrow (0, \infty)$ is any smooth function. In fact, we know from [4] that the radial projection is a harmonic morphisms and on the other hand, on can check that the function $f = \alpha(|x|)$ is positive and has vertical gradient.

Using the property of f-harmonic morphisms and Sacks-Uhlenberg's well-known result on the existence of harmonic 2-spheres we have the following proposition which gives many examples of f-harmonic maps from Euclidean 3-space into a manifold whose universal covering space is not contractible.

Proposition 2.10. *For any Riemannian manifold whose universal covering space is not contractible, there exists an f-harmonic map $\phi : \left(\mathbb{R}^3,\ ds_0^2\right) \longrightarrow (N^n, h)$ from Euclidean 3-space with $f(x) = \frac{2}{1+|x|^2}$.*

Proof. Let ds_0^2 denote the Euclidean metric on $\mathbb{R}^3$. It is well known that we can use the inverse of the stereographic projection to identify $\left(\mathbb{R}^3,\ \frac{4ds_0^2}{(1+|x|^2)^2}\right)$ with $S^3 \setminus \{N\} = \{(u_1, u_2, u_3, u_4) | \sum_{i=1}^4 u_i^2 = 1,\ u_4 \neq 1\}$, the Euclidean 3-sphere minus the north pole. In fact, the identification is given by the isometry

$$\sigma : \left(\mathbb{R}^3,\ \frac{4ds_0^2}{(1 + |x|^2)^2}\right) \longrightarrow S^3 \setminus \{N\} \subseteq \mathbb{R}^4$$

with $\sigma(x_1, x_2, x_3) = (\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{2x_3}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2})$. One can check that under this identification, the Hopf fiberation $\phi : \left(\mathbb{R}^3,\ \frac{4ds_0^2}{(1+|x|^2)^2}\right) \cong S^3 \setminus \{N\} \longrightarrow S^2$ can be written as

$$\phi(x_1, x_2, x_3) = (|z|^2 - |w|^2, 2zw),$$

where $z = \frac{2x_1}{1+|x|^2} + i\frac{2x_2}{1+|x|^2},\ \ w = \frac{2x_3}{1+|x|^2} + i\frac{|x|^2-1}{1+|x|^2}$. It is well known (see, e.g., [4]) that the Hopf fiberation ϕ is a harmonic morphism with dilation $\lambda = 2$. So, by Corollary 1.1, $\phi : \left(\mathbb{R}^3,\ ds_0^2\right) \longrightarrow S^2$ is an f-harmonic map with $f = \frac{2}{1+|x|^2}$. It is easy to see that this map is also horizontally conformal submersion and hence, by Theorem 2.2, it is an f-harmonic morphism. On the other hand, by a well-known result of Sacks-Uhlenbeck's, we know that there is a harmonic map $\rho : S^2 \longrightarrow$

(N^n, h) from 2-sphere into a manifold whose covering space is not contractible. It follows from Proposition 2.6 that the composition $\rho\circ\phi : \left(\mathbb{R}^3,\ ds_0^2\right) \longrightarrow (N^n, h)$ is an f-harmonic map with $f = \frac{2}{1+|x|^2}$. $\square$

Remark 1. We notice that the authors in [8] and [14] used heat flow method to study the existence of f-harmonic maps from closed unit disk $D^2 \longrightarrow S^2$ sending boundary to a single point. The f-harmonic morphism $\phi : \left(\mathbb{R}^3,\ ds_0^2\right) \longrightarrow S^2$ in Proposition 2.10 clearly restrict to an f-harmonic map $\phi : (D^3,\ ds_0^2) \longrightarrow S^2$ from 3-dimensional open disk into S^2. It would be interesting to know if there is any f-harmonic map from higher dimensional closed disk into two sphere. Though we know that $\phi : (M^m, g) \longrightarrow (N^n, h)$ being f-harmonic implies $\phi : (M^m, f^{\frac{2}{m-2}} g) \longrightarrow (N^n, h)$ being harmonic we need to be careful trying to use results from harmonic maps theory since a conformal change of metric may change the curvature and the completeness of the original manifold (M^m, g).

As we remark in Example 5 that any harmonic morphism is an f-harmonic morphism provided f is positive with vertical gradient, however, such a function need not always exist as the following proposition shows.

Proposition 2.11. *A Riemannian submersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ from non-negatively curved compact manifold with minimal fibers is an f-harmonic morphism if and only if $f = C > 0$. In particular, there exists no nonconstant positive function on S^{2n+1} so that the Hopf fiberation $\phi : S^{2n+1} \longrightarrow (N^n, h)$ is an f-harmonic morphism.*

Proof. By Corollary 2.8, a Riemannian submersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ with minimal fibers is an f-harmonic morphism if and only if $\operatorname{grad}\ln f$ is vertical, i.e., $\mathrm{d}\phi(\operatorname{grad}\ln f) = 0$. This, together with the following lemma will complete the proof of the proposition.

Lemma: Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be any Riemannian submersion of a compact positively curved manifold M. Then, there exists no (nonconstant) function $f : M \longrightarrow \mathbb{R}$ such that $\mathrm{d}\phi(\operatorname{grad}\ln f = 0)$.

Proof of the Lemma: Suppose $f : (M^m, g) \longrightarrow \mathbb{R}$ has vertical gradient. Consider

$$\left(M, e^{\varepsilon f} g\right)$$

where $\varepsilon > 0$ is a sufficiently small constant.

If ε is small enough, then $e^{2\varepsilon f} g$ is positively curved. One can check that

$$\phi : \left(M, e^{2\varepsilon f} g\right) \longrightarrow (N, h) \tag{14}$$

is a horizontally homothetic submersion with dilation $\lambda^2 = e^{-2\varepsilon f}$ since f has vertical gradient. By the main theorem in [24] we conclude that the map ϕ defined in (14) is a Riemannian submersion, which implies that the dilation and hence the function f has to be a constant. □

Remark 2. It would be very interesting to know if there exists any f-harmonic morphism (or f-harmonic map) $\phi : S^{2n+1} \longrightarrow (N^n, h)$ with non-constant f. Note that for the case of $n = 2$, the problem of classifying all f-harmonic morphisms $\phi : (S^3, g_0) \longrightarrow (N^2, h)$ (where g_0 denotes the standard Euclidean metric on the 3-sphere) amounts to classifying all harmonic morphisms $\phi : (S^3, f^2 g_0) \longrightarrow (N^2, h)$ from conformally flat 3-spheres. A partial result on the latter problem was given in [13] in which the author proves that a submersive harmonic morphism $\phi : (S^3, f^2 g_0) \longrightarrow (N^2, h)$ with non-vanishing horizontal curvature is the Hopf fiberation up to an isometry of (S^3, g_0). This implies that there exists no submersive f-harmonic morphism $\phi : (S^3, g_0) \longrightarrow (N^2, h)$ with non-constant f and the horizontal curvature $K_{\mathcal{H}}(f^2 g_0) \neq 0$.

Proposition 2.12. *For $m > n \geq 2$, a polynomial map (i.e. a map whose component functions are polynomials) $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an f-harmonic morphism if and only if ϕ is a harmonic morphism and f has vertical gradient.*

Proof. Let $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a polynomial map (i.e. a map whose component functions are polynomials). If ϕ is an f-harmonic morphism, then, by Theorem 2.2, it is a horizontally weakly conformal f-harmonic map. It was proved in [1] that any horizontally weakly conformal polynomial map between Euclidean spaces has to be harmonic. This implies that ϕ is also a harmonic morphism, and in this case we have $\mathrm{d}\phi(\operatorname{grad} f) = 0$ from (1). □

Example 6. $\phi : \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C}$ with $\phi(t, z) = p(z)$, where $p(z)$ is any polynomial function in z, is an f-harmonic morphism with $f(t, z) = \alpha(t)$ for any positive smooth function α.

Finally, we would like to point out that our notion of f-harmonic morphisms should not be confused with h-harmonic morphisms studied in [11] and [3], where an h-harmonic function is defined to be a solution of $\Delta\, u + 2g(\operatorname{grad} \ln h), \operatorname{grad}(u)) = 0$ (or equivalently, $h\Delta\, u + 2\mathrm{d}u(\operatorname{grad} h) = 0$), and an h-harmonic morphism is a continuous map between Riemannian manifolds which pulls back local harmonic functions to h-harmonic functions.

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References

[1] R. Ababou, P. Baird, and J. Brossard, *Polynômes semi-conformes et morphismes harmoniques* Math. Z. 231 (1999), no. 3, 589–604.

[2] M. Ara, *Geometry of F-harmonic maps* Kodai Math. J. 22 (1999), no. 2, 243–263.

[3] P. Baird and S. Gudmundsson, *p-Harmonic maps and minimal submanifolds*, Math. Ann., 294 (1992), 611-624.

[4] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press (2003).

[5] N. Course, *f-harmonic maps*, Thesis, University of Warwick, Coventry, CV4 7AL, UK, 2004.

[6] J. Cieśliński, A. Sym, and W. Wesselius, *On the geometry of the inhomogeneous Heisenberg ferromagnet: non-integrable case*, J. Phys. A. Math. Gen. 26 (1993) 1353-1364.

[7] J. Cieśliński, P. Goldstein, and A. Sym, *On integrability of the inhomogeneous Heisenberg ferromagnet model: Examination of a new test*, J. Phys. A: Math. Gen., 1994, 27: 1645-1664.

[8] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. 13 (2007), 423-435.

[9] M. Daniel, K. Porsezian, and M. Lakshmanan, *On the integrability of the inhomogeneous spherically symmetric Heisenberg ferromagnet in arbitrary dimension*, J. Math Phys, 1994, 35(10): 64986510.

[10] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1-68.

[11] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), vol **28** (1978), 107-144.

[12] S. Gudmundsson, *The geometry of harmonic morphisms*, Ph. D. Thesis, University of Leeds, UK, 1992.

[13] S. Heller, *Harmonic morphisms on conformally flat 3-spheres*, Bull. London Math. Soc., doi:10.1112/blms/bdq089, to appear.

[14] P. Huang and H. Tang, *On the heat flow of f-harmonic maps from D^2 into S^2*, Nonlinear Anal. 67 (2007), no. 7, 21492156.

[15] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., 19(2) (1979), 215-229.

[16] M. Lakshmanan and R. K. Bullough, *Geometry of generalised nonlinear Schrödinger and Heisenberg ferromagnetic spin equations with x-dependent coefficients*, Phys. Lett. A, 1980, 80(4): 287292.

[17] Y. X. Li and Y. D. Wang, *Bubbling location for f-harmonic maps and inhomogeneous Landau-Lifshitz equations*, Comment. Math. Helv. 81 (2) (2006) 433448.

[18] A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Symposia Mathematica III, Academic Press, London, 1970, pp. 341–402.

[19] E. Loubeau, *On p-harmonic morphisms*, Diff. Geom. and its appl., 12(2000), 219-229.

[20] J. Manfredi and V. Vespri, *n-harmonic morphisms in space are Möbius transformations*, Michigan Math. J. 41 (1994), 135142.

[21] Y.-L. Ou, *p-Harmonic morphisms, minimal foliations, and rigidity of metrics*, J. Geom. Phys. 52 (4) (2004) 365381.

[22] Y.-L. Ou, *On p-harmonic morphisms and conformally flat Spaces*, Math. Proc. Camb. Phil. Soc. (2005), 139-317.

[23] Y.-L. Ou, *p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps*, J. of Geo. Phy, Vol. 56, No. 3, 2006, 358-374.

[24] Y.-L. Ou and F. Wilhelm, *Horizontally homothetic submersions and nonnegative curvature.* Indiana Univ. Math. J. 56 (2007), no. 1, 243261.

[25] S. Ouakkas, R. Nasri, and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol. 10 (1), (2010), 11-27.

[26] H. Takeuchi, *Some conformal properties of p-harmonic maps and regularity for sphere-valued p-harmonic maps* , J. Math. Soc. Japan, 46(1994), 217-234.

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